CHAPTER 7

COUNTING POVERTY ORDERINGS AND DEPRIVATION CURVES

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ABSTRACT

Purpose – *A counting approach based on the number of deprivations suffered by the poor is quite an appropriate framework to measure multidimensional poverty with ordinal or categorical data. A method to identify the poor and a number of poverty indices have been proposed to take this framework into account. The implementation of this methodology involves the choice of a minimum number of deprivations required in order for an individual to be identified as poor. This cutoff and the choice of a poverty measure to aggregate the data are two sources of arbitrariness in poverty comparisons. The aim of this chapter is twofold. We first explore properties that characterize an identification method which allows different weights for different dimensions. Then the chapter examines dominance conditions in order to guarantee unanimous poverty rankings in a counting framework.*

Design/methodology/approach – *In the unidimensional poverty field, one branch of the literature is devoted to establishing dominance criteria that guarantee unanimous orderings at a variety of poverty thresholds and indices. This chapter takes this literature as a starting point, and investigates circumstances in which these ordering conditions may be applied in a weighted counting framework.*

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Findings – *Necessary and sufficient conditions are obtained that guarantee that two vectors, which represent the weighted sum of the deprivations felt by each person, may be unanimously ranked regardless of the identification cutoff and of the poverty measure.*

Originality/value – *Since most of the data available for measuring capabilities or dimensions of poverty is either ordinal or categorical, the counting approach provides an alternative framework that suits these types of variables. The implementation of the ordering conditions derived in this chapter is based on simple graphical devices that we call* dimension deprivation curves. *These curves become a useful way to check the robustness of poverty rankings to changes in the identification cutoff. They also provide a tool for determining nonambiguous poverty rankings in a wide set of multidimensional poverty indices that suit ordinal and categorical data.*

1. INTRODUCTION

In recent years there has been considerable agreement that poverty is a multidimensional phenomenon and great efforts have been made from both a theoretical and an empirical point of view, trying to assess multidimensional poverty.[1] Following Sen (1976), poverty measurement should consist of a method to identify the poor and an aggregative measure.

In a multidimensional framework, the identification of the poor usually incorporates two cutoffs. The first has to do with the traditional identification of the poor within each dimension using a dimension-specific poverty line. In the second step, a minimum number of deprived dimensions is required to be considered as a poor person. Thus a person is identified as poor if deprived in at least a given number of dimensions. The two extreme cases are referred to as the "union" and "intersection" approaches, respectively. Whereas the union procedure identifies someone who is deprived in at least one dimension as poor, the intersection definition requires a poor person to be deprived in all dimensions.

Nevertheless, how to identify the multidimensional poor is still a debatable issue and it may be of interest to explore properties to be fulfilled in this step. Allowing different weights for different dimensions, Section 2 explores properties of the second stage of the identification procedure. Once the poor people are identified, the *multidimensional headcount ratio*, which is the percentage of poor people in the society, may be gauged.

As regards the aggregation step, the majority of the indices introduced for the measurement of multidimensional poverty behave well only with cardinal variables, that is, with dimensions that are quantitative in nature. However, most of the data available to measure capabilities or dimensions of poverty are either ordinal or categorical. Consequently, only indices that deal well with qualitative variables should be used in empirical applications when this sort of data are involved.

What Atkinson (2003) refers to as a counting approach focuses on the number of dimensions in which each person is deprived, and is an appropriate procedure that deals well with ordinal and categorical variables. Among others, Chakravarty and D'Ambrosio (2006), Bossert, D'Ambrosio, and Peragine (2007), Alkire and Foster (2007), and Bossert, Chakravarty, and D'Ambrosio (2009) propose indices based on a counting approach. Specifically, Chakravarty and D'Ambrosio (2006) introduce the possibility that the dimensions may be weighted differently and derive counting measures in the social exclusion field. We also assume a weighted counting framework. In turn, Alkire and Foster (2007) propose what they call the *adjusted headcount ratio*, defined as the average of the number of deprivations suffered by the poor. The *multidimensional headcount ratio* and the *adjusted headcount ratio*, appropriately modified to incorporate dimensional weights, will play an important role in this work.

In general, the choice of either the identification cutoffs or the indices adds arbitrariness to poverty comparisons, and different selections can lead to contradictory results. For this reason it may be of interest to investigate conditions to guarantee that comparisons be unanimous to the different choices. There exists a branch of the literature devoted to establishing dominance criteria that provide unanimous orderings when comparisons are made at a variety of poverty thresholds and measures. Zheng (2000) provides a comprehensive survey of dominance conditions in the poverty unidimensional field. In this chapter, we take this literature as a starting point, and more specifically the papers by Shorrocks (1983), Foster (1985), and Foster and Shorrocks (1988a, 1988b). In particular, we investigate circumstances in which two vectors, which represent the number of weighted deprivations felt by each person, may be unanimously ranked regardless of the identification cutoff and of the poverty measure. In Section 3, we will show that if the ranking provided by the multidimensional headcount ratio is unambiguous over all admissible identification thresholds, then agreement is guaranteed over all counting poverty measures that satisfy the dimensional monotonicity property.[2] A similar result is obtained with respect to the (weighted) adjusted headcount ratio: rankings provided by this latter index are equivalent to agreement over all counting measures that fulfill monotonicity and distribution sensitivity. These results are by no means surprising. Atkinson (1987) derives a similar conclusion with regard to the headcount ratio in the unidimensional poverty field. In turn, Foster and Shorrocks (1988a, 1988b) characterize the poverty orderings obtained from the Foster–Greer–Thobercke measures (Foster, Greer, & Thorbecke (1984)).

The implementation of these conditions is based on two different types of curves that we call *dimension deprivation curves*, introduced in Section 3. The first one, which we call the *FD* curve, represents the multidimensional headcount ratio for all the admissible dimension cutoffs.[3] The second type of curve, henceforth *SD* curves, represents in the same picture the *headcount ratio*, the *adjusted headcount ratio*, and the *average deprivation share* according to the proposal by Alkire and Foster (2007), appropriately modified to incorporate dimensional weights.

Since the Lorenz curve was introduced in the literature, a number of cumulative curves have been widely used to check unanimous orderings in the inequality, poverty, and polarization fields.[4] In this connection, we will show that the curves proposed in this chapter become a powerful tool for checking unanimous orderings according to a wide class of counting measures. They also avoid the choice of an arbitrary identification cutoff and offer a useful way to determine the bounds of the number of dimensions for which multidimensional comparisons are robust. As the multidimensional headcount ratio and the adjusted headcount ratio behave particularly well with ordinal and categorical data, the *dimension deprivation curves* play a key role in making poverty comparisons when data are ordinal. The chapter finishes with some concluding remarks.

2. A WEIGHTED COUNTING POVERTY APPROACH

2.1. Notation and Basic Definitions

We consider a population of $n \geq 2$ individuals endowed with a bundle of $k \geq 2$ attributes considered as relevant to poverty measurement. The number of dimensions is given and fixed.

Regarding the identification of the poor through the specification of a poverty line, let us consider $z_j > 0$ to be the minimum quantity of the jth attribute for a subsistence level. An individual i is *deprived* as regards attribute j if $x_{ij} < z_j$.

In a counting approach, poverty is measured by taking into consideration the number of dimensions in which people are deprived. Thus we assume that the dimensions are represented by binary variables and characteristics of individual i are identified by a *deprivation vector* $\boldsymbol{x}_i \in \{0,1\}^k$, whose typical component j is defined by $x_{ij} = 1$ when individual i is deprived in attribute j and $x_{ij} = 0$ otherwise.

Allowing one dimension to be more important than another, let $\boldsymbol{w} \in \mathbb{R}^k_{++}$ be a vector of weights summing to k, whose jth component, w_j is the weight assigned to attribute j. We assume that the vector of weights is given and fixed.

Let us denote by d_i the weighted sum of the dimensions in which person i is deprived, that is, $d_i = \sum_{1 \le j \le d} w_j x_{ij}$, which represents the *poverty score* of individual i.[5] Let $D \subset [0,k]$ be the set of all admissible scores. In general, D is a discrete set of [0, k] containing the value 0, which corresponds to a person nondeprived in any dimension, and the value k, when the person is deprived in all the dimensions. When all weights are equal to 1 (i.e., all dimensions are assumed to be equally important), $D = \{0, 1, \ldots, k\}$. By contrast, if the weights are all different, then D is a discrete set with 2^k elements.

The vector $\boldsymbol{d} = (d_1, \ldots, d_n) \in D^n$ is referred to as the *vector of weighted deprivation counts*. This vector plays an important role in poverty measurement when ordinal data are involved. In fact this vector is invariant if the achievement levels and the poverty lines are transformed under the same monotonic transformations, and this is a crucial property when the achievements or capabilities are measured with ordinal variables. We will denote by $\bar{\boldsymbol{d}}$ the permutation of $\boldsymbol{d}$ in which individuals' poverty scores have been arranged in decreasing order, that is, $\bar{d}_i \ge \bar{d}_{i+1}$ for $i = 1, \ldots, n$. Hence people are ranked from the most deprived to the least. Let $G = \cup_{n \ge 1} D^n$ be the set of all admissible vectors of deprivation counts.

We will say that the vector $\boldsymbol{d}'$ is obtained from the vector $\boldsymbol{d}$ by a *permutation* if $\bar{\boldsymbol{d}}' = \bar{\boldsymbol{d}}$; by a *replication* if $\boldsymbol{d}' = (d, d, \ldots, d)$; by an *increment* (*decrement*) if $d'_i > d_i$ ($d'_i < d_i$) for some individual i, and $d'_j = d_j$ for all $j \ne i$.

2.2. The Identification of the Poor

The first step in measuring poverty is to identify the poor people. Two main methods have been used in this stage in a multidimensional setting, referred to as the "union" and the "intersection" approaches, respectively.

Whereas the union procedure identifies as poor someone who is deprived in at least one dimension, the intersection definition requires a poor person to be deprived in all dimensions. These methods present well-known drawbacks when the number of poverty dimensions is great. Whereas "almost nobody" is identified as poor with the intersection approach, "almost everybody" is poor with the union identification.

There is an intermediate procedure that proposes establishing a cutoff in the number of dimensions. If a weighted sum of deprivations is assigned to each person, this score may be used in the identification step. Thus, a person is identified as poor if the number of weighted dimensions in which they are deprived is at least m, that is, $d_i \geq m$, with $0 < m \leq k$. Person i is nonpoor otherwise, that is, if $d_i < m$.[6] For $m = \min_{1 \leq j \leq k} w_j$, this method coincides with the union approach, whereas for $m = k$, it is equivalent to the intersection approach. We will use ρ_m to denote this procedure. In this framework the identification function is assumed to be the same for all individuals.

The ρ_m method is simple and intuitive, and it may be worth examining the conditions that lead to a ρ_m identification method in a multidimensional setting.

For doing so, first of all we assume that the function that identifies the poor satisfies a property of *dichotomization*. This is a strong property that makes sense in a counting approach since it assumes that identifying a person as poor depends only on each individual's deprivation vector. This property is formalized as follows:

Dichotomization: An identification procedure ρ is a *dichotomized identification* function if $\rho : \{0,1\}^k \rightarrow \{0,1\}$ links $\boldsymbol{x}_i$, the vector of deprivations of individual i, with an indicator variable such that $\rho(\boldsymbol{x}_i) = 1$ if person i is identified as poor and $\rho(\boldsymbol{x}_i) = 0$ if person i is not poor.

In addition, we introduce a property for a dichotomized identification function. We think that a reasonable assumption is to require that if a person is considered as poor according to an identification method, then any other person deprived in equal or more weighted dimensions should also be considered as poor. We call this property *Poverty Consistency* and it is formulated as follows:

Poverty Consistency. Let ρ be a dichotomized identification function. We say that ρ satisfies the *poverty consistency property* if given a person i with $\rho(\boldsymbol{x}_i) = 1$ then $\rho(\boldsymbol{x}_{i'}) = 1$ for all person i' such that $d_i = \sum_{1 \leq j \leq d} w_j x_{ij} \leq d_{i'} = \sum_{1 \leq j \leq d} w_j x_{i'j}$.

The following proposition characterizes the ρ_m identification method.

Proposition 1. A nontrivial dichotomized identification function ρ fulfils the poverty consistency property if and only if there exists some $m \in (0, k]$ such that $\rho(\boldsymbol{x}_i) = 1$ if $d_i = \sum_{1 \leq j \leq d} w_j x_{ij} \geq m$ and $\rho(x_i) = 0$ otherwise.

Proof. Since ρ is a nontrivial identification method, there exists a person i such that $\rho(\boldsymbol{x}_i) = 1$. Let $m = \min_{i'=1,\ldots,n}\{d_{i'}/\rho(x_i) = 1\}$. For definition $\rho(\boldsymbol{x}_{i'}) = 0$ if $d_{i'} < m$ and for poverty consistency, $\rho(\boldsymbol{x}_{i'}) = 1$ if $d_{i'} \geq m$. The sufficiency of the proof is clear. Q.E.D.

As a consequence of this proposition, the only dichotomized identification method that is poverty consistent is ρ_m for some $m \in (0, k]$. Throughout this chapter the poor are supposed to be identified according to a ρ_m procedure.

Let us denote by Q_m and q_m, respectively, the set and number of poor identified using the dimension cutoff m. For each vector $\boldsymbol{d}$ of weighted deprivation counts, we define the *censored vector of deprivation counts*, denoted by $\boldsymbol{d}(\boldsymbol{m})$, as follows: $d_i(m) = d_i$ if $d_i \geq m$, and $d_i(m) = 0$ if $d_i < m$.

2.3. Aggregating Deprivations with a Counting Measure

The second step in poverty measurement is the aggregation of the poverty scores of the poor people. In what follows, a counting poverty measure P is a nonconstant function whose typical image, denoted by $P_m(\boldsymbol{d})$, represents the level of poverty in a society with a vector of weighted deprivation counts $\boldsymbol{d}$ and where the poor are identified according to a ρ_m procedure. The following four properties are the counterparts for a counting measure of the basic properties assumed in the poverty field.

First of all, since poverty measurement is concerned with poor people, it is usually demanded that a poverty index should not change under the improvements of the nonpoor people. In a counting approach, improvements are reflected in a decrease in the number of the weighted deprivations. Then, the poverty focus property may be formulated as follows.

Poverty Focus (PF): For any $m \in (0, k]$, P_m remains unchanged if the poverty score of a nonpoor person decreases.

It may be worth noting that PF ensures that $P_m(\boldsymbol{d}) = P_m[\boldsymbol{d}(\boldsymbol{m})]$.

In order to narrow down the shape of the measure, it may be interesting to identify some types of transformations that seem to have an effect on the poverty level, and to require the index to be consistent with them.

If one believes that all the individuals and all the dimensions are essential in measuring poverty, then it appears intuitive to demand that if the poverty score of any poor individual decreases, then the overall poverty should decrease. Note that the decrement in the poverty score of a poor person may lift them out of poverty.[7] This property may be is formulated as follows:

Dimensional Monotonicity (MON): For any $m \in (0,k]$, $P_m(\boldsymbol{d}') < P_m(\boldsymbol{d})$ if $d'_i < d_i$ for some individual i with $d_i \geq m$, and $d'_j = d_j$ for all $j \neq i$.

According to Sen (1976), a poverty measure should be sensitive to distribution among the poor and greater weight should be attached to the poorer person. Consequently, a decrease in poverty due to a decrease in the poverty score of a poor person should be greater the higher the score of the person is. Let us consider two poor individuals, i and j, such that the poverty score of individual i is higher than that of individual j, that is, $d_i > d_j$. Let us assume that it is possible to decrease the two scores by the same amount. MON ensures that the poverty level decreases under the two transformations. Nevertheless, the next axiom goes beyond MON and demands that the decrease in poverty under the former decrement (that of the poverty score of the poorer), should be higher than that under the latter. As in MON, the two individuals involved in the transformation are allowed to lift out of poverty.[8] This axiom may be formalized as follows.

Distribution Sensitivity (DS): For any $m \in (0,k]$ and $h > 0$:

$$P_m(\boldsymbol{d}) - P_m(d_1, \ldots, d_i - h, \ldots, d_j, \ldots, d_n) > P_m(\boldsymbol{d}) - P_m(d_1, \ldots, d_i, \ldots, d_j - h, \ldots, d_n)$$

if $(d_1, \ldots, d_i - h, \ldots, d_j, \ldots, d_n), (d_1, \ldots, d_i, \ldots, d_j - h, \ldots, d_n) \in D^n$ and $d_i > d_j \geq m$.

The following property establishes that no other characteristic apart from the number of weighted dimensions in which a person is deprived matters in defining a counting poverty index. This principle is much stronger than its counterpart in the unidimensional field since it implicitly entails a trade-off between the dimensions. For instance, when all the dimensions are weighted equally this property implies that it does not matter in which particular dimensions people are deprived and, somehow, all of them become interchangeable. A similar conclusion may be obtained if the weights are different.

Symmetry (SYM): For all $m \in (0,k]$, $P_m(\boldsymbol{d}) = P_m(\boldsymbol{d}')$ if $\bar{\boldsymbol{d}}' = \bar{\boldsymbol{d}}$.

Finally, the following condition allows the comparisons of populations of different sizes.

Replication Invariance (RI): For all $m \in (0,k]$, $P_m(\boldsymbol{d}) = P_m(\boldsymbol{d}')$ if $\boldsymbol{d}' = (\boldsymbol{d}, \boldsymbol{d}, \ldots, \boldsymbol{d})$.

We define the following two inclusive classes of counting poverty measures:

$$\mathbf{P}_1 = \{P \textit{ counting poverty measure}/P \textit{ satisfies } PF, MON, SYM \textit{ and } RI\}$$

$$\mathbf{P}_2 = \{P \textit{ counting poverty measure}/P \textit{ satisfies } PF, MON, DS, SYM \textit{ and } RI\}$$

Clearly $\mathbf{P}_2 \subset \mathbf{P}_1$, and as will be shown, the inclusion is strict.

The first counting poverty measure introduced in the literature is the *multidimensional headcount ratio*, denoted by H, which is the percentage of poor people in the society. In other words, for each $m \in (0,k]$ identification cutoff, $H_m = q_m/n$ is the percentage of the population whose poverty scores are higher than or equal to m. This index, usually used to measure the incidence of poverty, is unable to capture the intensity, is not distribution sensitive and violates MON, that is, it does not change if a person already identified as poor becomes deprived in an additional dimension.

The *(weighted) adjusted headcount ratio,* M, is defined as the ratio of the number of weighted deprivations suffered by the poor to the total number of weighted deprivations, that is, $M_m(\boldsymbol{d}) = [\sum_{1 \le i \le n} d_i(m)]/nk$. In contrast to the headcount ratio this index satisfies MON, although it does not belong to class $\mathbf{P}_2$ since it violates DS.

More information about poverty can be incorporated by using the *weighted average deprivation share across the poor* denoted by A, introduced by Alkire and Foster (2007) when all the dimensions are equally weighted. This is defined as the mean among the poor, of the weighted sum of the deprivations suffered by the poor, that is, $A_m(\boldsymbol{d}) = [\sum_{1 \le i \le n} d_i(m)]/q_m k$. This index captures the intensity of poverty. Furthermore, it holds that $M_m(\boldsymbol{d}) = H_m A_m(\boldsymbol{d})$.

3. COUNTING POVERTY ORDERINGS

This section is concerned about how to rank two vectors of weighted deprivation counts in order to evaluate whether poverty is higher in one society than in another. Poverty rankings may be reversed depending on the identification threshold, or on the measure selected. Thus, in order to avoid contradictory results, poverty orderings require unanimous rankings for a set of identification cutoffs, or a class of poverty measures. As it is impossible to check unanimity for infinite pairwise comparisons, ordering conditions are derived to characterize unanimous agreement. Following the literature, given a counting poverty measure, P, we define the *partial ordering with respect to P*, denoted by $\precsim_P$, in the set of vectors of deprivation counts, by the rule[9]

$$\boldsymbol{d}' \precsim_P \boldsymbol{d} \text{ if and only if } P_m(\boldsymbol{d}') \geq P_m(\boldsymbol{d}) \text{ for all } m \in (0, k]$$

In this section, we will examine the partial poverty orderings with respect to the *multidimensional headcount ratio*, H, and the *weighted adjusted headcount ratio*, M.

3.1. Poverty Ordering with Respect to the Multidimensional Headcount Ratio, H*, and the* FD *Curve*

As already mentioned, given a cutoff of the number of dimensions, m, the multidimensional headcount ratio, H_m, gauges the percentage of poor people according to the identification procedure ρ_m. For any vector of deprivation counts it is possible to consider the graph of H as a function of this dimension cutoff, ranked in decreasing order. We will refer to this curve as the *FD* curve associated with the vector $\boldsymbol{d}$, and its ordinates are computed as follows:

$$FD(\boldsymbol{d}; p) = H_{k-p}, p \in [0, k]$$

The following example helps to clarify this. Let us consider a society of 10 individuals endowed with 4 attributes weighted equally. Let us assume that the vector of deprivation counts is $\boldsymbol{d} = (4, 3, 3, 2, 2, 1, 1, 1, 0, 0)$. The *FD* curve for this vector is displayed in Fig. 1.

Some interesting properties of this curve may be mentioned. The *FD* curve is an increasing step function that is right-continuous. The horizontal axis displays the identification cutoffs ranked in decreasing order, and in the vertical axis, by definition, the multidimensional headcount ratio, H_m, is recovered. Two limiting curves correspond with the extreme situations: if nobody is deprived, the curve coincides with the horizontal axis; whereas, if everybody is deprived in all dimensions, the curve becomes the parallel line to the horizontal axis through the point (0,1).

It is clear from the graph that, for two vectors of deprivation counts with the same population size, $\boldsymbol{d}, \boldsymbol{d}' \in D^n$, if $\boldsymbol{d}' \precsim_H \boldsymbol{d}$, that is, if $H_m(\boldsymbol{d}') \geq H_m(\boldsymbol{d})$ for all $m \in (0, k]$, then the *FD* curve of $\boldsymbol{d}$ must be below or to the left of the

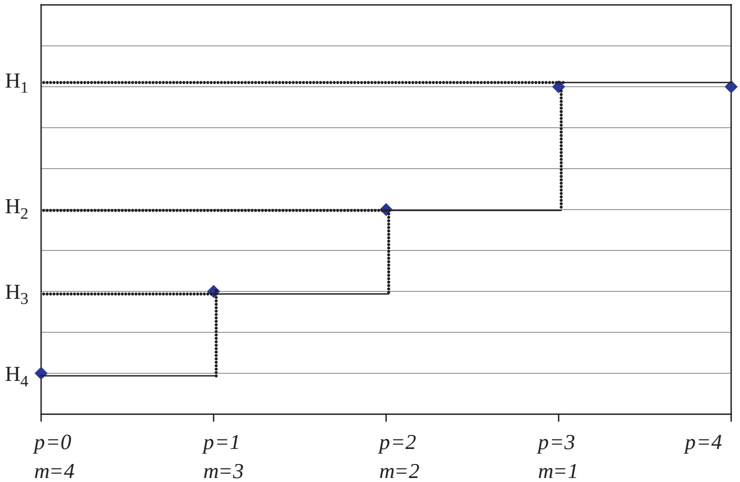

Fig. 1. Plotting the Identification Cutoffs and the Headcount Ratio: The *FD* Curve.

FD curve of $\boldsymbol{d}'$. And the reverse is also true: if the *FD* curve of $\boldsymbol{d}$ is below or to the left of the *FD* curve of $\boldsymbol{d}'$ then $H_m(\boldsymbol{d}') \geq H_m(\boldsymbol{d})$ for all $m \in (0, k]$, and $\boldsymbol{d}' \precsim_H \boldsymbol{d}$. We get the following proposition.

Proposition 2. For any $\boldsymbol{d}, \boldsymbol{d}' \in D^n$ vectors of deprivation counts, the following statements are equivalent:

(i) $FD(\boldsymbol{d}; p) \leq FD(\boldsymbol{d}'; p)$ for all $p \in [0, k]$;
(ii) $H_m(\boldsymbol{d}) \leq H_m(\boldsymbol{d}')$ for all $m \in (0, k]$;
(iii) $\bar{d}_i \leq \bar{d}'_i$ for all $i = 1, \ldots, n$;
(iv) $\bar{\boldsymbol{d}}'$ may be obtained from $\bar{\boldsymbol{d}}$ by a finite sequence of increments;
(v) $\sum_{1 \leq i \leq n} \varphi(\bar{d}'_i) \leq \sum_{1 \leq i \leq n} \varphi(\bar{d}'_i)$ for all continuous, increasing functions $\varphi : [0, k] \rightarrow \mathbb{R}$

Proof. In the appendix.

This proposition shows that when the *FD* curve of a vector of deprivation counts lies above or to the right of the curve of other with the same population size, or equivalently, when these two vectors can be ordered with respect to *H*, then one may be obtained from the other by a finite sequence of permutations and/or increments. Consequently, any poverty measure belonging to class $\mathbf{P}_1$ will rank these two vectors exactly in the same way. Moreover, since both *H* and the measures belonging to class $\mathbf{P}_1$ are replication invariant, the result also holds for vectors with different population sizes.

The reverse is also true. In fact, consider the following class of counting measures: $P(\boldsymbol{d}, m) = 1/n\{\sum_{1 \le i \le n} \psi[d_i(m)]\}$, with $\psi : [0, k] \to \mathbb{R}$, a continuous strictly increasing convex function. It is quite simple to show that P belongs to class $\mathbf{P}_1$. Given any continuous increasing function $\varphi : [0, k] \to \mathbb{R}$ and $\varepsilon > 0$, then the measures $P_\varepsilon(\boldsymbol{d}, m) = 1/n[\sum_{1 \le i \le n}(\varepsilon\psi + \varphi)(d_i(m))]$ also belong to class $\mathbf{P}_1$. Consequently, given two vectors $\boldsymbol{d}$ and $\boldsymbol{d}'$ with $P_\varepsilon(\boldsymbol{d}, m) \le P_\varepsilon(\boldsymbol{d}', m)$, when $\varepsilon \to 0$ we get statement (v) in Proposition 2 and have the following result, that links the ordering with respect to H with first-degree stochastic dominance:

Proposition 3. For any $\boldsymbol{d}, \boldsymbol{d}' \in G$ vectors of weighted deprivation counts:

$$FD(\boldsymbol{d}'; p) \ge FD(\boldsymbol{d}; p) \text{ for all } p \in [0, k]$$

if and only if $P_m(\boldsymbol{d}') \ge P_m(\boldsymbol{d})$ for all $P \in \mathbf{P_1}$ and for all identification cutoff $m \in (0, k]$.

This proposition reveals that, although H fails to satisfy MON, the ordering with respect to H is equivalent to agreement over all counting measures satisfying MON. Consequently, if the *FD* curves of two vectors of deprivation counts do not intersect, then all poverty counting measures satisfying MON will lead to the same verdict.

By contrast, when the curves intersect, there are two possibilities in order to obtain unanimous ranking: either the set of measures is restricted, as shown in Section 2.2, or the admissible cutoffs are limited, as will be developed in Section 2.3.

3.2. Poverty Ordering with Respect to the Adjusted Headcount Ratio, M*, and the* SD *Curve*

One interesting feature of the *FD* curve introduced in the previous section is that, given a vector $\boldsymbol{d}$ and a dimension threshold m, it is straightforward to prove that the area beneath the curve of the censored vector, $FD[\boldsymbol{d}(\boldsymbol{m})]$, is equal to $d\,M_m$. Thus, even if a conclusive poverty verdict could not be reached with the H ordering, it would be possible to get unanimous rankings with respect to M.

As usual, we propose constructing the *SD* curve, for any vector $\boldsymbol{d}$, plotting the headcount ratio against the adjusted headcount ratio, that is, pairs of points (H_m, M_m). We also plot two extreme points $(0, 0)$ as the start of the curve, and $(1, M_1)$, as the end of the curve. Then we join the dots. Fig. 2 shows the *SD* curve associated with the vector $\boldsymbol{d}$ in the previous example.

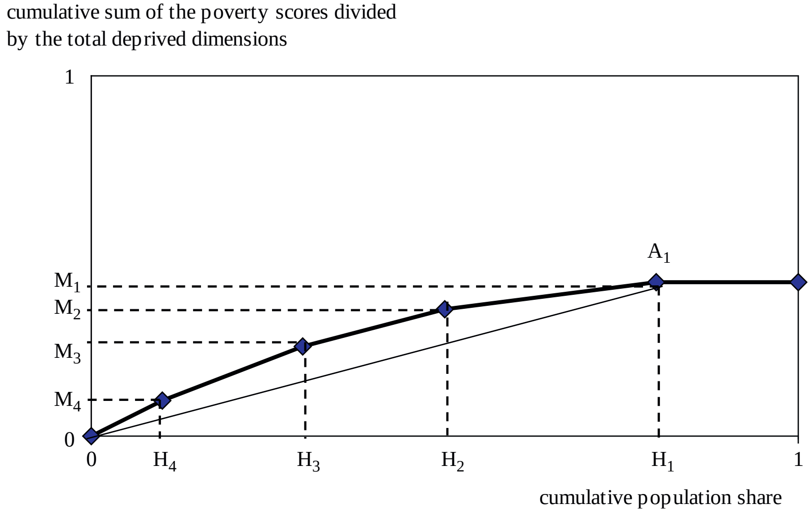

Fig. 2. Plotting the Headcount Ratio and the Adjusted Headcount Ratio: The *SD* Curve.

It may be worth noting that for any vector of deprivation counts $\boldsymbol{d}$, ranked from the highest poverty score to the lowest, the *SD* curve can equivalently be defined in the following way: for each integer $p = 0, \ldots, n-1$ the ordinate of the curve is computed as the cumulative of the sum of poverty scores of the first p people divided by the total number of deprivations that could possibly be experienced by all people. At intermediate points the curve is determined by linear interpolation. Thus, the ordinates of the *SD* curve are computed as follows:

$$SD(\boldsymbol{d};0) = 0$$

$$SD\left(\boldsymbol{d};\frac{p}{n}\right) = \frac{1}{nk}\sum_{1\le i\le p} d_i, p = 1,\ldots,n$$

$$SD\left(\boldsymbol{d};\frac{p+\theta}{n}\right) = \frac{1}{nk}\sum_{1\le i\le p}(d_i + \theta d_{p+1}), p = 0,\ldots,n-1, \theta \in [0,1]$$

The ordinates of this curve are replication invariant, and are also invariant to permutations of $\boldsymbol{d}$. The graph, as displayed in Fig. 2, begins at the origin, and is a continuous nondecreasing concave function.

There are two boundaries that correspond to the extreme situations of minimum and maximum deprivation. If nobody is deprived, the curve coincides with the horizontal axis. By contrast, if everybody is deprived in all dimensions, the curve becomes the diagonal line.

By construction, on the vertical axis the curve shows the percentage of the weighted sum of deprivations experienced by the percentage of the most deprived population, which is displayed on the horizontal axis. Thus, the maximum value of the curve corresponds to the percentage of the deprivations experienced by all the people, that is, the adjusted headcount ratio according to the union procedure. The point at which the curve becomes horizontal yields the percentage of people deprived in at least one dimension: the headcount ratio according to the union procedure.

Less obvious, perhaps, is the meaning of each point at which the slope of the curve changes. It should be noted that when people with the same poverty scores are accumulated, the slope does not change. By contrast, adding a person whose poverty score is less than the existing ones makes the slope decrease. Thus, each of the points at which the slope changes, yields the percentage of people deprived in at least, say, m weighted dimensions, or H_m. The vertical axis on the other hand displays, by definition, the adjusted headcount ratio M_m. For instance, the first point at which the slope changes shows the percentage of deprived dimensions suffered by the population deprived in all dimensions. In other words, according to the intersection procedure, the headcount ratio (on the horizontal axis) and the adjusted headcount ratio (on the vertical axis) are recovered at this point.

The *weighted average deprivation share across the poor*, A_m, is also represented in the graph by the slope of the ray from (0,0) to $[p, SD(p)]$.

The following proposition is based on the results established by Marshall and Olkin (1979, Propositions 4.A.2 and A.B.2) for vectors with the same number of components:

Proposition 4. For any $\boldsymbol{d}, \boldsymbol{d}' \in D^n$ vectors of deprivation counts, the following statements are equivalent:

(i) $SD(\boldsymbol{d}'; p) \geq SD(\boldsymbol{d}; p)$ for all $p \in [0, 1]$;
(ii) $M_m(\boldsymbol{d}) \leq M_m(\boldsymbol{d}')$ for all $m \in (0, k]$;
(iii) $\sum_{1 \leq i \leq p} \bar{\bar{d}}_i \leq \sum_{1 \leq i \leq p} \bar{\bar{d}}'_i$ for all $p = 1, \ldots, n$;
(iv) $\boldsymbol{d}'$ may be obtained from $\boldsymbol{d}$ by a finite sequence of permutations, increments and/or transformations of the form $T(\boldsymbol{z}) = (z_1, \ldots, z_i + h, \ldots, z_j - h, \ldots, z_n)$ with $h > 0$ and $z_i \geq z_j$;
(v) $\sum_{1 \leq i \leq n} \varphi(\bar{d}_i) \leq \sum_{1 \leq i \leq n} \varphi(\bar{d}'_i)$ for all continuous, increasing and convex functions $\varphi : [0, k] \to \mathbb{R}$.

Proof. In the appendix.

An implication of this proposition is the result below.

Proposition 5. For any $\boldsymbol{d}, \boldsymbol{d}' \in D^n$ vectors of deprivation counts and for any measure $P \in \mathbf{P}_2$, if $SD(\boldsymbol{d}'; p) \geq SD(\boldsymbol{d}; p)$ for all $p \in [0, 1]$ then $P_m(\boldsymbol{d}') \geq P_m(\boldsymbol{d})$ for all $m \in (0, k]$.

Proof. In the appendix.

Consequently, when the *SD* curve of a vector $\boldsymbol{d}'$ lies above the curve of another, $\boldsymbol{d}$, with the same population size, any poverty measure belonging to class $\mathbf{P}_2$ will rank these two vectors in exactly the same way. In addition, as both the deprivation curves and the measures $P \in \mathbf{P}_2$ are invariant under replication, the result also holds for vectors with different population sizes. The reverse is also true and the proof is completely similar to the corresponding result in the previous section. So we get:

Proposition 6. For any $\boldsymbol{d}, \boldsymbol{d}' \in G$ vectors of deprivation counts:

$$SD(\boldsymbol{d}'; p) \geq SD(\boldsymbol{d}; p) \text{ for all } p \in [0, 1]$$

if and only if $P_m(\boldsymbol{d}') \geq P_m(\boldsymbol{d})$ for all $P \in \mathbf{P_2}$ and for all identification cutoff $m \in (0, k]$.

Then, this result reveals that although M, the dimension adjusted headcount ratio, violates DS, if two vectors of deprivation counts can be unanimously ranked by M_m at all dimension cutoffs, then all poverty counting measures satisfying DS will rank societies in the same way.

3.3. Poverty Ordering when the Curves Intersect

When the dimension deprivation curves introduced in the two previous sections intersect, it is still possible to establish dominance conditions by restricting the set of identification cutoffs. In fact, even if the curves of two vectors cross, there exists a threshold $m^* \in (0, k]$ that corresponds with the identification cutoff after which the intersection occurs. In other words, m^* ensures that the curves do not intersect for all $m \in (m^*, k]$, which becomes the relevant set for the cutoffs. A simple way to establish dominance conditions in these cases is to base comparisons on the censored vectors, and to modify the results derived in the previous sections accordingly. Taking into consideration the respective censored vectors, denoted by $\boldsymbol{d}(m^*)$ and $\boldsymbol{d}'(m^*)$, we get the following proposition.

Proposition 7. For any $\boldsymbol{d}, \boldsymbol{d}' \in G$ vectors of deprivation counts:

(i) $FD(\boldsymbol{d}'(\boldsymbol{m}^*); p) \geq FD(\boldsymbol{d}(\boldsymbol{m}^*); p)$ for all $p \in [0, k]$

if and only if $P_m(\boldsymbol{d}') \geq P_m(\boldsymbol{d})$ for all $P \in \mathbf{P}_1$ and for all identification cutoff $m \in (m^*, k]$.

(ii) $SD[\boldsymbol{d}'(\boldsymbol{m}^*); p] \geq SD[\boldsymbol{d}(\boldsymbol{m}^*); p]$ for all $p \in [0, 1]$

if and only if $P_m(\boldsymbol{d}') \geq P_m(\boldsymbol{d})$ for all $P \in \mathbf{P}_2$ and for all identification cutoff $m \in (m^*, k]$.

The implication of this proposition is that, even when the *dimension deprivation* curves intersect, they allow us to obtain robust conclusions in a wide set of counting measures restricting the set of identification cutoffs. Since not all the admissible cutoffs are equally meaningful in poverty measurement, this result may be quite useful in empirical applications: when two deprivation vectors cannot be unanimous ranked for all cutoffs, concentrating on the poorest people can lead to conclusive verdict.

4. CONCLUDING REMARKS

A counting approach that concentrates on the number of dimensions in which each person is deprived is an appropriate procedure to measure multidimensional poverty with ordinal and categorical variables.

The choice of a cutoff to identify the poor, and a poverty measure to aggregate the data are two sources of arbitrariness and different selections may lead to contradictory conclusions. In this chapter we have characterized the identification procedure and have derived dominance conditions in order to obtain unanimous rankings in a wide set of counting measures, and a set of identification cutoffs.

The implementation of these conditions is based on two different types of *dimension deprivation* curves, which guarantee unanimous rankings of vectors of deprivation counts when they do not intersect. And, even if the curves cross, additional results are derived that lead to conclusive verdicts by restricting the admissible cutoffs in the identification of the poor. Thus, these curves become a useful way to determine the boundaries of the number of dimensions for which counting poverty comparisons are robust and have been shown to play a key role in making poverty comparisons when the data are ordinal.

Policy makers should choose the dimension-specific poverty line and the weight attached to any dimension.

NOTES

1. A comprehensive survey on multidimensional poverty can be found in Chakravarty (2009).
2. This property demands that the measure should decrease if the number of dimensions, in which a poor person is deprived, decreases.
3. This curve is quite similar to the *deprivation distribution profile* proposed by Subramanian (2009). However, two main differences can be pointed out. On the one hand, we propose to represent this cumulative curve as a step function that is right-continuous. On the other hand, to our knowledge, S. Subramanian does not derive dominance conditions in his paper.
4. Among them the TIP curves proposed by Jenkins and Lambert (1997), the poverty curves in Foster and Shorrocks (1988b), the polarization curve introduced by Foster and Wolfson (2010), and the proposal of Shorrocks (2009) to derive unemployment indices.
5. Bossert et al. (2009) characterize this first stage of aggregation of the characteristics of each individual.
6. Assuming that all the dimensions are equally weighted, this intermediate identification method is followed by Mack and Lansley (1985), Gordon, Nandy, Pantazis, Pemberton, and Townsend (2003), and Alkire and Foster (2007) among others.
7. This is the counterpart of what Donaldson and Weymark (1986) refers to as the strong monotonicity axiom in the unidimensional poverty field. Zheng (1997) discusses different types of monotonicity axioms and their relationships.
8. This principle has the same spirit as the *Transfer Sensitivity Axiom* introduced by Sen (1976) as long as progressive transfers among the poverty scores make sense. A discussion about the relationship between these axioms may be found in Zheng (1997).
9. We follow Atkinson (1987) and adopt the weak definition of a partial ordering. Although not all the results derived in this chapter hold for the other two levels (the semistrict and the strict ones), similar conditions could be also obtained in these two cases.

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APPENDIX

Proof of Proposition 2. The equivalence between (i) and (ii) follows from the definition of *FD* curve. To prove that (ii) implies (iii), let us suppose that there exists $i \in \{1, \ldots, n\}$ such that $\bar{d}_j \leq \bar{d}'_j$ for all $j = 1, \ldots, i-1$ and $\bar{d}_i > \bar{d}'_i$. Taking $m = d_i$, we find that $H_m(\boldsymbol{d}) > H_m(\boldsymbol{d}')$. That (iii) implies (ii) is clear. After (iii), writing $\bar{d}'_i = \bar{d}_i + (\bar{d}'_i - \bar{d}_i)$ we have (iv). That (iv) implies (v) is straightforward. If (v) holds, since the function $\varphi_1(z) = \max\{z - \bar{d}'_1, 0\}$ is continuous and increasing we find that $\bar{d}_1 \leq \bar{d}'_1$. Let us suppose that there exists $i \in \{1, \ldots, n\}$ such that $\bar{d}_j \leq \bar{d}'_j$ for all $j = 1, \ldots, i-1$ and $\bar{d}_i > \bar{d}'_i$. If we consider the continuous increasing function

$$\varphi^*(z) = \begin{cases} 1 & \text{if } z \geq \bar{d}_i \\ (z - \bar{d}'_i)/(\bar{d}_i - \bar{d}'_i) & \text{if } \bar{d}'_i < z < \bar{d}_i \\ 0 & \text{if } z \leq \bar{d}'_i \end{cases},$$

we find that $\sum_{1 \leq i \leq n} \varphi^*(\bar{d}_i) \geq i > \sum_{1 \leq i \leq n} \varphi^*(\bar{d}'_i) = i - 1$. Then (v) implies (iii) and the proof is complete. Q.E.D.

To prove Proposition 4 we use the following result established by Marshall and Olkin (1979, propositions 4.A.2 and A.B.2):

Lemma 1. For any $\boldsymbol{d}, \boldsymbol{d}' \in \mathbb{R}^n_+$ the following conditions are equivalent:

(i) $\sum_{1 \leq i \leq p} \bar{d}_i \leq \sum_{1 \leq i \leq p} \bar{d}'_i$ for all $p = 1, \ldots, n$;
(ii) $\boldsymbol{d}$ may be obtained from $\boldsymbol{d}'$ by successive applications of a finite number of T transforms of the form $T_1(\boldsymbol{z}) = T_1[z_1, \ldots, z_{i-1}, \lambda z_i + (1-\lambda) z_j, z_{i+1}, \ldots, z_{j-1}, \lambda z_j + (1-\lambda) z_i, z_{j+1}, \ldots, z_n]$ where $0 \leq \lambda \leq 1$; and/or of the form $T_2(\boldsymbol{z}) = T_2(z_1, \ldots, z_{i-1}, \alpha z_i, z_{i+1}, \ldots, z_n)$, where $0 \leq \alpha < 1$;
(iii) $\sum_{1 \leq i \leq n} \varphi(\bar{d}_i) \leq \sum_{1 \leq i \leq n} \varphi(\bar{d}'_i)$ for all continuous, increasing and convex functions $\varphi : [0, k] \to \mathbb{R}$.

Proof of Proposition 4. From the definitions of *SD* curve and the weighted adjusted headcount ratio, it is clear that (i), (ii), and (iii) are equivalent. From Lemma 1, (iii) is also equivalent to (v).

Moreover, according to (ii) in Lemma 1, under the same hypothesis, $\boldsymbol{d}$ may be obtained from $\boldsymbol{d}'$ by $\boldsymbol{d} = T_2(\boldsymbol{d}')$, that is, by a decrement; and/or by $\boldsymbol{d} = T_1(\boldsymbol{d}')$. Note that for $\lambda = 0$, T_1 reduces to a permutation. For the rest of values of λ, as permutations are allowed, we may assume, without loss of generality that $z_i > z_j$ and $\lambda \in [1/2, 1]$. Defining $h = (1 - \lambda)(z_i - z_j) > 0$, T_1 may rewritten as $T_1(\boldsymbol{z}) = T_1(z_1, \ldots, z_i - h, \ldots, z_j + h, \ldots, z_n)$, with $z_i - h \geq z_j + h$, and T_1 is the inverse of the T transformation of (iv). Q.E.D.

Proof of Proposition 5. From Proposition 4 (iv) holds. Then it is enough to prove that if $\boldsymbol{d}' = (d_1, \ldots, d_i + h, \ldots, d_j - h, \ldots, d_n)$ with $h > 0$ and $d_i \geq d_j$ then for any $m \in (0, k]$, $P_m(\boldsymbol{d}') \geq P_m(\boldsymbol{d})$. If $d_j \geq m$, $P_m(\boldsymbol{d}') > P_m(\boldsymbol{d})$ since P_m satisfies DS. If $d_i + h \geq m > d_j$, $P_m(\boldsymbol{d}') > P_m(\boldsymbol{d})$ by MON. Otherwise, $P_m(\boldsymbol{d}') = P_m(\boldsymbol{d})$ by PF. Q.E.D.